AN ATTEMPT TO HANDLE THE CALCULUS PROBLEM

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ABSTRACT

For many years now, we have experienced a negative trend concerning the mathematics skill level of our new engineering students. This has been particularly noticeable in the calculus course, which is mandatory for all engineering programs at Jönköping University. To handle this problem, we have made some structural changes to the way we teach the subject, emphasizing the teaching efforts to standard type problems – named 'A-problems'. A list of categories of such A-problems have been constructed, the purpose of which is to help the student identify the most important ideas and develop the basic skills needed to understand calculus. The students are also given the option to form groups and solve selected problems together and then hand in their solutions for marking. If a group have presented enough correct solutions this grants the group members some bonus points for the final exam, which is divided into two parts where the first one deals entirely with A-problems. During the past few years this way of teaching calculus have been implemented by all math teachers and we are cautiously optimistic concerning the results.

KEYWORDS

Mathematics curriculum, teaching methods, examination

INTRODUCTION

Teaching calculus is a challenge. Maybe it always was like this but these days it can seem like an almost insurmountable task to teach a student how to differentiate a composite function, not to speak of trying to make the student grasp the concept of differentiation in itself. People tend to like challenges though, and this is no exception; teaching calculus, has to be admitted, is also a lot of fun! There are several reasons that makes it particularly challenging today though and we will mention a few here.

Firstly we have to acknowledge the large variability of the skill level our engineering students have in pre university mathematics. This means that some students (unfortunately not many) have quite good technical skills and also understand rather abstract ideas right from start while others may have a hard time adding fractions (I kid you not). The majority of the students are somewhere in between these two extremes of course, but the 'skill distribution density' is not symmetric – it has a centre of mass on the lower end of the scale. It also seems that this centre of mass is moving in the wrong direction with time i.e. average Joe knows less today than his older sister did a few years back. This is an unfortunate trend which has been observed in Sweden (as well as in many other western developed countries) for quite many years now. Recently, the results from a large comparative study of the skill level in mandatory school mathematics of 15 year olds called PISA [1] not only confirms this but also showed that relatively speaking, Sweden is losing ground compared with other similar countries. We had the worst result of the Nordic countries and a mediocre result compared to other European countries. The education system in Sweden is now under heavy scrutiny and a major revision is going to be implemented at pre university level (one idea mentioned in the papers even involded sending out special teams of “elite” teachers to problem schools) and we can only hope that this will change things for the better as swiftly as possible.

Add to the above the fact that mathematics in general, and calculus in particular, is a highly cumulative subject and it stands to reason that we face some non trivial problems when meeting the students in our classrooms.

THE SETUP IN JÖNKÖPING

At Jönköping University all engineering students have to take a basic calculus course which corresponds to 7.5 ECTS-credits, or an eighth part of an academic year in Sweden. The course is taught during a period of 8 weeks and then there is a written exam. Usually there are about 100 students per course and the typical teacher is an experienced lecturer with some background in the research field of mathematical analysis. The teaching methods are comprised of traditional lectures (full class) combined with tutorial sessions in smaller groups.

The curriculum of the calculus course is fairly standard (at least from a Swedish perspective); we start out by introducing the basic properties of the real numbers and we end the course by solving some simple ordinary differential equations. A summary of the topics covered follows below.

- Elementary logic and set theory
- Number systems including complex numbers
- Equations and inequalities
- Elementary functions; definitions and properties
- Limits
- Continuity
- Differentiation
- Integration
- Ordinary differential equations of 1’st and 2’nd order

Since this is the only calculus course many of our engineering students will ever take, we have made the choice to include everything which is necessary to understand the concept of a differential equation, but very little on top of this. For example, we do not have the time to cover numerical methods more than superficially or to even mention Taylor series which is unfortunate. Some of our students study these topics later in a multi variable calculus course though.

The book used [2] is published only a few years ago but the material is presented in an old fashioned rather rigorous style, there are many similar Swedish books on the market and most university calculus courses uses a book of this type. In our experience, many of the students have a hard time *reading* mathematics by themselves; instead they tend to skip the theory pages (proofs are omitted by default) and jump directly to the exercises, where they then get stuck and call for the teacher to explain things. We believe this to be a behaviour induced from the earlier school systems where mathematics is taught in a very non-theoretical fashion based on repeating exercises at best; the concepts of a theorem and a proof are carefully avoided. The skill of actually *reading* mathematics, following a chain of written down arguments leading to some conclusion is not developed at all. Therefore there tend to be something of a cultural shock for many students when meeting mathematics at university level where the (good) text books usually are loaded with rigorous definitions, theorems and proofs.

In search of the holy grail

As should be clear from the introduction, mathematics teachers in many countries working at university level have had reasons to seek ways to improve the student results for quite a few years now. Unfortunately, but not surprisingly, no quick fix has been found although a number of different types of educational reforms have been tried, especially in the primary school systems. For example, in the U.S. there has been a rather infected debate going on for a few decades, known as the math wars, about the pre university mathematics education involving (the majority of) researchers of mathematics on the "traditionalist" side and (the majority of) researchers of mathematics education on the other "reform" side [3]. In a nutshell, the reform side want to focus less on computational skills and more on conceptual understanding and exploration, the traditionalists being not so impressed by these ideas. It's interesting that in Sweden we also have had a similar version of this "war", with similar groups of people involved on both sides debating similar ideas of reform. Initially the reform side got a lot of attention and many schools tried the suggested new ways of teaching, but now it seems the pendulum is starting to swing back again; the results from the experiments have not been encouraging and going back to the recent PISA study [1], we can see that many of the countries with the best results are also the countries with the most traditional ways of teaching, emphasizing algorithms, algebra and drill type exercises.

The thoughts leading to our reformed calculus course

In Jönköping, going back to how the situation was before the year of 2000, a typical calculus course would get great student reviews which was always a boost for the teacher involved, but at the same time it was a depressing affair to mark the final exam, since a normal result would mean failing roughly 50% of the students. The written exam was constructed in a very traditional way; 8-10 problems was more or less randomly constructed with the aim of testing as many of the learning outcomes as possible. The problems was worth a total of 25 points and in order to pass the course, a student would need to score at least 10 points, and for the higher grades the limits were set to 15 and 20 points respectively.

We knew from experience that at Jönköping university, a typical student that follows one of our 3-year engineering educations at bachelor level does not see mathematics as a very interesting subject in itself, it is only a necessary means for becoming an engineer. Very often the student have very little positive experience of mathematics from earlier school systems, it can be quite the opposite. We mathematicians try to motivate the students as best we know how but it's not so easy. Recently we have tried to invite a professor of mechanics as a guest lecturer to speak of the necessity of mathematics showing some nice examples. This has been appreciated by the students and we will continue to investigate further ideas as far as motivation go.

Optional bonus point program

So if the students have

- poor initial mathematical skills
- the attitude that almost everything in life is more interesting than mathematics

what should be done? Furthermore, we also had access to statistics indicating that a student is only willing to spend an average of 30 hours per week on school studies, and we realized that there should be a potential for improvement here – we wanted the students to spend more time per week doing mathematics. Trying to accomplish this we chose the way of the carrot rather than the stick and a few years back we therefore invented an optional bonus point program that the students may enrol in. Basically, they have to form study groups of 4 people and hand in solutions to weekly assignments consisting of a number of problems dealing with theory recently discussed in the classroom. We wanted to encourage cooperation and catalyze mathematical discussions so we only demanded one solution per study group. We mark their solutions and hand it back to them with feedback. If they get approximately 80% correct solutions then they will get a number of bonus points for the final exam. The maximum number of bonus points granted is 3 and only valid for scores below 10 (i.e. for the lowest grades). For the higher grades the maximum number of bonus points granted is 2 and 1 respectively.

After implementing this we noticed that the activity during the tutorial sessions increased and we also noticed study groups spending time with these hand in problems during hours not formally scheduled. In some case we even got complains from other teachers that the students were spending too much time doing mathematics, it "stole" time from their course! Not all effects were positive though, we noticed that some students tend to focus almost all their attention to the hand in problems, not doing the regular exercises from the course book. Also, since this is a group task, we can never be sure that all students participate in an equal fashion. Never the less, the bonus point program is still in place today and we see no way back now, we believe the advantages outweighs the few negative effects. One of our teachers have also developed a quite sophisticated system with scripts that automatically generate an arbitrary number of variations of a selected problem so even if there are 100 students, every single one gets individual hand in problems.

THE A-PROBLEMS

The above described bonus point program was implemented in Jönköping after the millennia shift, and afterwards we noticed that the student activity had increased somewhat and the results were slightly improved but not dramatically; we still regularly had final exams where almost 50% would fail. In a way this was now even more depressing because this meant that every other student had scored less than 7 points (out of 25) on the final exam. It should be mentioned that due to rationalization demands during these years the classes that took calculus increased in size; in ten years we went from a typical class size of 50 students to the situation today where you usually have between 100 and 150 students. Obviously this have had some negative effects on the results; at the very least we experience that today we have a larger proportion of students with very weak skills in pre university mathematics compared to the situation 10 years ago.

So, once again we realized that we had to do something. As a teacher group we agreed that a written mandatory final exam is a must in this type of course and we didn't want to change the examination method to a system where almost everything is based on hand in projects or something similar although this could lead to a quick increase in the number of passed students. Such systems are in place at several universities in Sweden but we consider them unfair and arbitrary and we are too much of traditionalists to even consider this as a viable examination system at our department.

We wanted an examination system which guaranteed that if we pass a student then this person would have the skills to independently solve a majority of the problem types we study in the course, let it be on a basic level. In other words, we wanted to honestly be able to claim that at least a majority of the learning outcomes were indeed fulfilled by this same student. What we came up with was the idea to construct a list of problem types which we now call the A-problems. Essentially, this is an interpretation of the learning outcomes from the course syllabus into concrete problems, or at least concrete problem types, more well defined than the somewhat general descriptions of the learning outcomes. Our hope and intent was that this list would be particularly helpful for the typical weak student when trying to grasp what kind of skills he or she is expected to learn from the calculus course. For a problem type to appear on this list we required that

1. the problem type should be a concrete example of a skill described as a learning outcome in the course syllabus
2. the mathematics involved in the problem type should be “new” to the student, i.e. the problem type should not have been covered in earlier math courses from pre university education.

The syntax used for each problem type when writing the list was to start with a description of the problem type, followed by one or more concrete examples of this problem type. For each problem type we also make references to the relevant exercises in the course book that deals with this particular problem type. This means that the list both defines what is considered an important skill and it also helps the student to develop this skill. Finally, we give the students two versions of the list; one exactly as described above, and another version where we have provided detailed solutions to our example problems.

A new final

The point of the A-problem categorization is made clear only when seen in the context of the final written exam. We wanted to be able to tell the students “If you learn how to solve all these different problem types which appears on the A-list then you will have no trouble passing the final exam”. In order to make the truth of this statement perfectly clear to the students we decided to change the form of the exam dividing it into two parts. The first part, called part A, deals *only* with A-problems and here it is possible to score a maximum of 15 points i.e. more than enough to pass the course (recall that the grade limits are 10/15/20 points respectively).

The second part of the exam, called part B, consists of more demanding problems worth a total of 10 points. Our ambition is to construct these B-problems in a way so that when solving these problems the student either has to combine several different skills learned from solving A-problems, or has to come up with some new ideas i.e. it is not enough to be able to repeat a standard algorithm, some creative reasoning is also necessary. This means that even the quite talented student can get some kicks out of this course, and it also has the upside that for the teacher this B-part is quite fun to construct (although it takes a lot of time; I can spend several days constructing one final exam). We also have the rule that we will only mark the part B if the student has scored at least 10 points on the part A, this in order to make sure that a passed student has learned enough basic skills which correspond to the learning outcomes of the course.

It should be mentioned that we do not allow our students to use calculators when writing the final exam. We have the experience that many students have developed a behaviour where they rely much too heavily on calculators in their earlier mathematics education, being able to solve some types of problems without having a clue what they really are doing mathematically speaking; essentially they just press some buttons and take whatever result the calculators produce as an undisputed “truth”. The pros and cons of using calculators is a huge topic that deserves a paper on its own so I will simply make this statement leave it at that for now.

EXAMPLES

The learning outcomes

Here I will list the learning outcomes from the course syllabus and hopefully it will later on be clear how these have been interpreted as A-problems.

After completion of the course the student should be able to

- perform simple calculations involving complex numbers.
- understand the concept of a function and especially know how the elementary functions behave (elementary functions means polynomials, trigonometric functions and their inverses and exponential and logarithm functions).
- solve basic equations and inequalities involving the elementary functions.
- understand the concept of a limit and be able to compute simple limits by using standard limits
- understand the concept of continuity and use the fundamental theorems valid for continuous functions
- formulate the definition of a derivative and understand its interpretation in various situations
- differentiate expressions involving the elementary functions by using the differentiation rules and to use the derivative as a tool when sketching graphs and for solving applied problems such as optimization problems
- compute simple primitive functions, definite integrals and generalized integrals
- solve linear and separable first order differential equations of and second order linear differential equations of second order with constant coefficients.

A selection of A-problems

I will now give a couple of examples of A-problems generated from the learning outcomes. Some A-problems we have derived directly from one single learning outcome while others take in aspects from several learning outcomes. After each problem type description I will also give concrete example problems of the corresponding type.

<u>A-problem:</u> Performing basic calculations involving complex numbers represented either in Cartesian or polar form. Solving equations of the form $z^n = w$ where n is a positive integer and w is a complex number.

- Let $z = \frac{3+4i}{1-i}$. Write z in the form $a + bi$ and compute $|z|$.
- Solve the equation $z(3+i) - 2i\bar{z} = 2$.
- The equation $z^4 - 3z^3 + 2z^2 + 2z - 4 = 0$ has one solution $z = 1 + i$. Find the other three solutions.
- Find all solutions to the equation $z^3 = 8i$. The answer should be in Cartesian form.

<u>A-problem</u>: Solving equations involving square roots.

- Solve the equation $\sqrt{2-x} = 2x - 1$.

<u>A-problem</u>: Solving equations and inequalities involving rational functions and the absolute value function by using a sign table or division into separate cases.

- Solve the inequality $\frac{x^2-10}{x-3} \leq 3$.
- Solve the inequality $|x-5| + |x+3| < 10$.

A-problem : Calculations involving elementary functions including computations of domains, ranges, inverse functions, derivatives.

- Let $f(x) = e^{2x}$ and $g(x) = \sqrt{x}$. Determine the functions $f(g(x)), g(f(x))$ and describe their domains and ranges.
- Determine whether or not the function $f(x) = \ln(x - x^2)$ is invertible.
- Compute the tangent line at $x = 2$ to the function $f(x) = \arctan(\frac{2}{x})$.

A-problem: Solving equations involving logarithms and exponential functions.

- Solve the equation $\ln(4x + 22) - 2\ln(x + 1) = 1$.

A-problem: Solving equations involving trigonometric and inverse trigonometric functions.

- Solve the equation $\arcsin(x) - \arccos(x) = \frac{\pi}{6}$.

A-problem: Computing derivatives using the definition of derivative.

- Use the definition of derivative to compute the derivative of $f(x) = \frac{1}{\sqrt{x}}$.

A-problem: Using the derivative as a tool for solving optimization problems and sketching the graph of a function.

- Sketch the graph of the function $f(x) = \frac{x^3}{x^2-1}$. Also find all local extreme values of this function and compute any asymptote that might exist.
- Find all extreme values of the function $f(x) = |x + 1|e^{-x/2}$ if the domain is set to be the interval $[-1,2]$.

A-problem: Finding primitive functions using the method of substitution.

- Compute $\int \sin(x)\sqrt{2\cos(x) + 1}dx$.

A-problem: Finding primitive functions using the method of integration by parts

- Compute $\int (x + 1)\ln(x)\,dx$.

A-problem: Solving linear and separable first order differential equations.

- Solve the differential equation $xy' + \frac{1}{2}y = x^2, x \geq 1, y(1) = 1$.
- Solve the differential equation $y' = y^2\cos(x)$.

A-problem: Solving second order linear differential equations with constant coefficients of the form $y'' + ay' + by = f(x)$. The function $f(x)$ is either a polynomial, a trigonometric function or of the form $p(x)e^{kx}$ where $p(x)$ is a polynomial.

- Find all solutions to the differential equation $y'' - 2y' - 3y = 2e^{3x}$.

Some of the B-problems

Since a B-problem can be pretty much anything, I just give a couple of examples of what has been given as exam problems so the reader get the flavor.

- Let $f(x) = \frac{1}{2}e^{2x} - 3e^{x} + 2x + 1$. Find the smallest number a such that $f(x)$ is invertible on the interval $[a, \infty]$. For this a, find the domain of the inverse function.
- If $f(x)$ is a continuous function such that $0 \leq f(x) \leq 1$, show that the equation $f(x) = x$ must have at least one solution on the interval $[0,1]$.
- Which is largest of the numbers $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots$
- Find the range of the function $f(x) = e^{x}\ln{(1 + e^{-x})}$.
- Find positive numbers $x_1, \ldots, x_{10}$ such that the expression $\frac{x_1}{x_{10}} + \frac{x_2}{x_9} + \cdots + \frac{x_{10}}{x_1}$ becomes as small as possible.
- The heating system suddenly stops in a house where the temperature initially is $20℃$. Assume the temperature gradient to be directly proportional to the difference between inner and outer temperature. If the temperature in the house after 2 hours is $15℃$, what is the temperature in the house after 24 hours?

CONCLUSIONS – THE GOOD, THE BAD AND THE UGLY

The good,

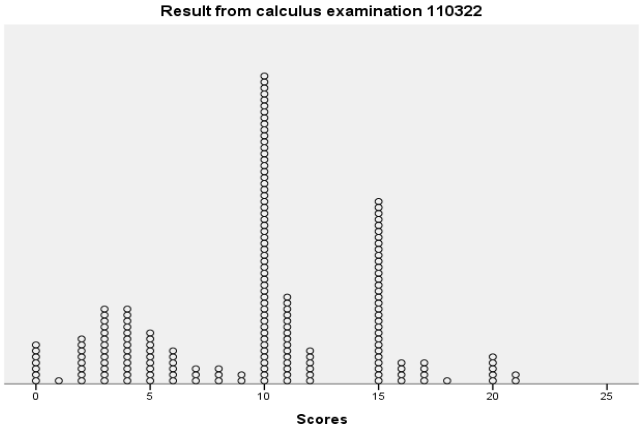
After this system has been in place now for a couple of years we have noticed a slight increase as far as the results from the final exam go. The increase is not very large though; in general some two thirds now pass the first final exam which is only slightly better than before. We have got mainly positive feedback from our students and they really seem to like this system where the question "what should I study in order to pass the exam" seems to have been answered once and for all.

the bad,

On the negative side we see that many of the students now focus entirely on the A-problems in order just to pass the exam (with the lowest grade) even though they many times have the potential to get higher grades. We also suspect that some students fail that today that would have passed with the old type of exam where they could try to solve more types of problems than today.

...and the ugly

It could be argued that although the results now are somewhat better when looking at the final exam, we are not so sure if the students that pass the calculus course today really *are* better than they used to be. Many of the students that pass the course today seem to have quite shallow understanding of sometimes very fundamental concepts in Calculus although they know how to solve a sufficient number of A-problems. Maybe in the future it would be good to try to incorporate more problems that tests conceptual understanding rather than the ability to learn and repeat algorithms. As always, the results we arrive at in the end of the course very much depend on *how* we try to measure the skill level of our students. I will include a typical but quite ugly picture of my latest final exam. Here 185 students took this exam and around 160 of these have followed the course seriously, participating in the bonus point program.



As can be seen some 120 students score 10 points or more which means that they pass. The reasons for the abnormal heights at 10,15 and 20 points is explained by the fact that the bonus points kick the score to 10, 15 or 20 if the actual score was in the range 7-9 points, 13-14 points or 19 points respectively. Apparently, quite a few of the students (that scored in the interval 7-9) have not managed to solve more than 3 of the A-problems correctly and therefore had to rely on the bonus points in order to pass the course which is a bit sad in the end.

The question still remains: how do we teach calculus to a large group of students in a very limited time with very limited resources? The answer is that we have no clue but we keep trying anyway.

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